DE GIORGI'S PROOF OF THE $C^{0,\alpha}$-REGULARITY THEOREM FOR ELLIPTIC EQUATIONS WITH MEASURABLE COEFFICIENTS

FRANCESCO MAGGI

Contents

1. Overview

The $C^{0,\alpha}$-regularity theorem for weak solutions to elliptic equations with measurable coefficients asserts that if Ω is an open set in $\mathbb{R}^n$ $(n \ge 2)$ and $u \in W^{1,2}_{\rm loc}(\Omega)$ is a weak solution of the elliptic equation $-\mathrm{div}\,(A(x)\nabla u) = 0$, in the sense that

$$\int_\Omega \nabla\varphi \cdot A(x)\,\nabla u = 0 \qquad \forall \varphi \in C^\infty_c(\Omega)\,, \tag{1.1}$$

then $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ for some $\alpha = \alpha(n, \Lambda/\lambda) \in (0,1)$. Here $A(x)$ is a measurable field of symmetric matrices which is uniformly elliptic, i.e., there exist constants $0 < \lambda \le \Lambda < \infty$ such that $\lambda\,\mathrm{id} \le A(x) \le \Lambda\,\mathrm{id}$ for a.e. $x \in \Omega$. The first proofs of this celebrated result are independently due to De Giorgi and Nash, in two historical and very influential papers. These lecture notes focus on De Giorgi's proof, and have been written for the *PDE I* course taught by the author at UT Austin during the Fall 2019 semester.

2. Isoperimetry and boundedness of isoperimetric sets

This section, informal concerning mathematical rigor, has the aim to illustrate how isoperimetric/Sobolev inequalities can be used to exploit energy minimality to obtain decay estimates for certain geometric quantities. Both the equivalence between isoperimetric and Sobolev inequalities (through the coarea formula), and the fact that these bounds could be combined with minimality to prove density/decay estimates, are two signature traits of the original point of view on elliptic equations introduced by De Giorgi in his work. These ideas take a particular simple form in the geometric setting of isoperimetric problems (at least if one is not interested in rigorous justifications), and seem to provide a clear motivation for the key argument used in proving the Boundedness Theorem, Theorem 3.2.

The setting for our toy, illustrative model, will be the following. We look at sets $E \subset \mathbb{R}^n$ with volume $|E|$ and perimeter $P(E)$, where $P(E)$ stands for "the $(n-1)$-dimensional measure of the boundary of E". We write $P(E) = \mathcal{H}^{n-1}(\partial E)$ with, say, $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$ and ∂E the topological boundary of E. This is not *exactly* the right technical setting, which indeed requires the introduction of sets of finite perimeter, but will serve well for our more limited (and more interesting!) illustrative purposes.

With this notation set, we consider an **isoperimetric set** E **in** $\mathbb{R}^n$, i.e.

$$P(E)\le P(F)\qquad \forall F\subset\mathbb{R}^n\,,|F|=|E|\,. \tag{2.1}$$

Everyone knows that such a set E must be a ball, but we will pretend to ignore it, and to set for ourself the task of showing that **every set** E **satisfying** (2.1) **has diameter bounded in terms of** n**,** $|E|$ **and** $P(E)$. We will prove this result by exploiting the validity of

$$P(E)\ge c(n)\,|E|^{(n-1)/n}\,, \tag{2.2}$$

for some constant $c(n)>0$. By combining Fubini's theorem and the divergence theorem we can easily show that (2.2) holds with $c(n)=1$ for example. This is not the sharp constant, and of course computing the sharp constant $c(n)=n\,|B_1|^{1/n}$ requires proving that the balls are isoperimetric sets (which is a stronger result than our diameter estimate). However, we shall only need (2.2) in non-sharp form to prove the diameter estimate, so that our argument could still be useful in an hypothetic approach to solving the isoperimetric problem in $\mathbb{R}^n$ which would require an a-priori diameter bound to conclude the sphericity of isoperimetric sets.

We now start proving the diameter estimate. We fix the direction e_1 and look into the volume function

$$V(t)=|E\cap\{x_1>t\}|\,.$$

We aim at showing that **if**

$$V(t_0)\le\varepsilon_0$$

for some universal ε_0 **depending on** n**,** $|E|$ **and** $|E|/P(E)$ **only (see** (2.12) **below), then we must have**

$$V(t_0+T_0)=0\qquad \text{where } T_0=4\,n\,V(t_0)^{1/n},$$

which, in particular, implies the one-sided bound $E\subset\{x_1<t_0+T_0\}$. This point of view of "conditional" boundedness estimate is used in two key parts of De Giorgi's proof of the $C^{0,\alpha}$-theorem.

To prove the decay of $V(t)$ we compare E with $E\cap\{x_1<t\}$, its truncation by the half-space $\{x_1<t\}$, by means of (2.1). Since $|E\cap\{x_1<t\}|<|E|$ (unless $V(t)=0$ and thus we have nothing to prove!) we cannot set $F=E\cap\{x_1<t\}$ in (2.1), but we rather have to work with

$$F=\lambda(t)\big(E\cap\{x_1<t\}\big)\,,$$

where $\lambda(t)$ is the scaling factor given by

$$\lambda(t)=\Big(\frac{|E|}{|E\cap\{x_1<t\}|}\Big)^{1/n}=\Big(\frac{|E|}{|E|-V(t)}\Big)^{1/n}\ge 1\,,$$

and, thus, such that $|F|=|E|$. Considering that $P(\lambda\,G)=\lambda^{n-1}\,P(G)$ for every $G\subset\mathbb{R}^n$, we deduce from (2.1) the family of inequalities

$$P(E)\le\Big(\frac{|E|}{|E|-V(t)}\Big)^{1/n}P(E\cap\{x_1<t\})\,,\qquad \forall t\in\mathbb{R}\,. \tag{2.3}$$

These are rewritten more conveniently by noticing that, for a generic value of t,

$$P(E)=\mathcal{H}^{n-1}(\{x_1<t\}\cap\partial E)+\mathcal{H}^{n-1}(\{x_1>t\}\cap\partial E)\,, \tag{2.4}$$

$$P(E\cap\{x_1<t\})=\mathcal{H}^{n-1}(\{x_1<t\}\cap\partial E)+\mathcal{H}^{n-1}(E\cap\{x_1=t\})\,. \tag{2.5}$$

Indeed these relations hold if ∂E has no vertical parts of positive area inside $\{x_1=t\}$, i.e. if $\mathcal{H}^{n-1}(\partial E\cap\{x_1=t\})=0$. This holds for at most countably many values of t, for otherwise we would have $P(E)=+\infty$ (a sum on non-negative terms parameterized over an uncountable set is finite only if at most countably many terms are positive). This is also a good point to take note of two additional geometric facts: the first one characterizes the second term on the RHS of (2.5) as the (negative) derivative of $V(t)$, i.e.

$$\mathcal{H}^{n-1}(E\cap\{x_1=t\})=-V'(t)\,,\qquad\text{for a.e. } t\,, \tag{2.6}$$

and it is immediately deduced by Fubini't theorem $V(t)=\int_{-\infty}^t \mathcal{H}^{n-1}(\{x_1=s\}\cap E)\,ds$; the second one is a direct consequence of the divergence theorem,

$$\mathcal{H}^{n-1}(E\cap\{x_1=t\})\le\mathcal{H}^{n-1}(\{x_1>t\}\cap\partial E)\,, \tag{2.7}$$

with equality only iff $V(t)=0$. Indeed, by applying the divergence theorem on the set $E\cap\{x_1>t\}$ with constant vector field e_1, and noticing that the boundary of $E\cap\{x_1>t\}$ consists of the union of $E\cap\{x_1=t\}$ (outer unit normal $-e_1$) and $\{x_1>t\}\cap\partial E$ (same outer unit normal ν_E as ∂E), we see that

$$\begin{aligned}0 &= \int_{E\cap\{x_1>t\}}\text{div } e_1=\int_{\partial(E\cap\{x_1>t\})} e_1\cdot\nu_{E\cap\{x_1>t\}}\\ &= \int_{E\cap\{x_1=t\}} e_1\cdot(-e_1)+\int_{\{x_1>t\}\cap\partial E}\nu_E\cdot e_1\,,\end{aligned}$$

so that (2.7) follows, with equality holds iff $\nu_E=e_1$ on $\{x_1>t\}\cap\partial E$, which is equivalent (by an additional argument) to $V(t)=0$. Notice that (2.7) combined with (2.4) and (2.5) also gives

$$P(E\cap\{x_1<t\})\le P(E)\,, \tag{2.8}$$

for a.e. $t\in\mathbb{R}$, with strict inequality unless $V(t)=0$.

By combining (2.4), (2.5), (2.6) and the Lipschitz bound

$$\Big(\frac{1}{1-a}\Big)^{(n-1)/n}\le 1+C(n)a\qquad\forall\alpha\in(0,1/2)\,,$$

we deduce from (2.3) that, if $V(t)\le|E|/2$, then

$$\begin{aligned}&\mathcal{H}^{n-1}(\{x_1<t\}\cap\partial E)+\mathcal{H}^{n-1}(\{x_1>t\}\cap\partial E)\\ &\le\Big(1+C(n)\,\frac{V(t)}{|E|}\Big)\Big\{\mathcal{H}^{n-1}(\{x_1<t\}\cap\partial E)-V'(t)\Big\}\,.\end{aligned}$$

By canceling out $\mathcal{H}^{n-1}(\{x_1<t\}\cap\partial E)$ on both sides, and then by recalling that the terms between $\{...\}$ equal $P(E\cap\{x_1<t\})\le P(E)$ (recall (2.8)), we thus find

$$\mathcal{H}^{n-1}(\{x_1>t\}\cap\partial E)\le -V'(t)+C(n)\,\frac{P(E)}{|E|}\,V(t)\;,\qquad\text{for a.e. } t \text{ s.t. } V(t)\le\tfrac{|E|}{2}\,. \tag{2.9}$$

We now wish to exploit the non-sharp isoperimetric inequality $P(G)\ge|G|^{(n-1)/n}$ to deduce from (2.3) a differential inequality (implying decay in finite time) for $V(t)$. To this end, we add to both side of (2.3) a term $-V'(t)=\mathcal{H}^{n-1}(E\cap\{x_1=t\})$ and then exploit the variant of (2.5) for $E\cap\{x_1>t\}$ to rewrite (2.9) into the equivalent form

$$P(\{x_1>t\}\cap E)\le -2\,V'(t)+C(n)\,\frac{P(E)}{|E|}\,V(t)\;,\qquad\text{for a.e. } t \text{ s.t. } V(t)\le\tfrac{|E|}{2}\,. \tag{2.10}$$

By applying $P(G)\ge|G|^{(n-1)/n}$ with $G=\{x_1>t\}\cap\partial E$ we thus find

$$V(t)^{(n-1)/n}\le -2\,V'(t)+C_0(n)\,\frac{P(E)}{|E|}\,V(t)\;,\qquad\text{for a.e. } t \text{ s.t. } V(t)\le\tfrac{|E|}{2}\,. \tag{2.11}$$

The key analytic feature brought in by isoperimetric inequality is the appearance of the **sublinear** power $(n-1)/n$ of $V(t)$ on the LHS of (2.11); indeed, the monotonicity of $V(t)$ and the fact that $V(t)$ appears with a linear power on the RHS of (2.11) will allow us to reabsorb the linear term on the RHS into the sublinear term on the LHS. More precisely, we notice that

$$C_0(n)\,\frac{P(E)}{|E|}\,V(t)\le\frac{1}{2}\,V(t)^{(n-1)/n}\qquad\text{iff}\qquad V(t)\le\Big(\frac{|E|}{2\,C(n)\,P(E)}\Big)^n\,.$$

Setting

$$\varepsilon_0=\min\Big\{\Big(\frac{|E|}{2\,C_0(n)\,P(E)}\Big)^n,\frac{|E|}{2}\Big\}\,,\tag{2.12}$$

we find that if $V(t_0)\le\varepsilon_0$ then $V(t)\le\varepsilon_0$ for every $t>t_0$ and thus

$$C_0(n)\,\frac{P(E)}{|E|}\,V(t)\le\frac{1}{2}\,V(t)^{(n-1)/n}\qquad\forall t>t_0\,,$$

allowing us to deduce from (2.11)

$$\frac{V(t)^{(n-1)/n}}{2}\le-2\,V'(t)\,,\qquad\text{for a.e. }t\in(t_0,\infty)\,.\tag{2.13}$$

This is easily rewritten as

$$\frac{1}{4}\le-\Big(n\,V(t)^{1/n}\Big)'\,,\qquad\text{for a.e. }t\in(t_0,\infty)\cap\{V>0\}\,,$$

which gives

$$V(t)^{1/n}\le V(t_0)^{1/n}-\frac{t-t_0}{4n}\qquad\forall t\in(t_0,\infty)\cap\{V>0\}\,.$$

In other words we have $V(t_0+T_0)=0$, and thus $E\subset\{x_1<t_0+T_0\}$, provided we set

$$T_0=4\,n\,V(t_0)^{1/n}\,.$$

This proves the conditional boundedness estimate we stated above. We notice (for the sake of completeness) that the conditional boundedness estimate has to be combined with some additional arguments in order to deduce the containment of E into a slab $\{a<x_1<b\}$ with $b-a$ bounded in terms of n, $P(E)$ and $|E|$. These arguments are omitted, as they are less relevant to the proof of the $C^{0,\alpha}$-theorem. A good reference for a complete discussion is found in [Mag08].

3. Boundedness for solutions to elliptic equations

We now consider $u\in W^{1,2}_{\rm loc}(\Omega)$ solving $-{\rm div}\,(A(x)\nabla u)=0$ in the sense of distributions, i.e.

$$\int\nabla\varphi\cdot A(x)\nabla u=0\qquad\forall\varphi\in C^\infty_c(\Omega)\,,$$

where $A(x)$ is uniformly elliptic as in the introduction. The goal of this section is proving that $u\in L^\infty_{\rm loc}(\Omega)$, but we shall actually prove an estimate which contains much more information, as it will give uniform upper (and lower) bounds on u in terms of weaker, integral quantities. The structure of the argument is the same one we have illustrated on the isoperimetric problem, and we shall use that analogy to guide our exposition. To this end we start noticing that a solution u to (1.1) is minimizing the energy

$$Q(u)=\int\nabla u\cdot A(x)\nabla u$$

with respect to compactly supported variations of u in Ω (and with the support of the variation intended as the domain of integration to take into account that we are not necessarily assuming $u\in W^{1,2}(\Omega)$). Having in the comparison argument from the previous section, where the perimeter of the set E was compared to that of its truncation $E\cap\{x_1<t\}$ (properly rescaled to fix the volume constraint), see (2.3) (or its alternative forms (2.9) and (2.10)), we now compare the Q-energy of u with its truncation $\min\{u,t\}$ (properly localized to fix the boundary data). Testing Q-minimality against a competitor is however

equivalent to test the elliptic equation (1.1) against a specific variation. In this case, $\varphi = \zeta^2\,(u-t)_+$ where ζ is a localizing factor. The usual proof of the Caccioppoli inequality is then repeated verbatim to find the following estimate: for every $B_R \subset\subset \Omega$, $r < R$, and $t \in \mathbb{R}$ we have

$$\int_{\{u>t\}\cap B_r} |\nabla u|^2 \le \frac{C(n,\Lambda)}{(R-r)^2} \int_{B_R} (u-t)_+^2\,. \tag{3.1}$$

Needless to say, an analogous estimate holds for truncations from below,

$$\int_{\{u<t\}\cap B_r} |\nabla u|^2 \le \frac{C(n,\Lambda)}{(R-r)^2} \int_{B_R} (t-u)_+^2\,. \tag{3.2}$$

Given these two families of inequalities, one does not need to invoke (1.1) anymore for proving that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$. For this reason, the following definition is introduced: given $\gamma > 0$, a function $u \in W^{1,2}_{\rm loc}(\Omega)$ is in the **upper De Giorgi class** $DG_\gamma^+(\Omega)$ if

$$\int_{\{u>t\}\cap B_r} |\nabla u|^2 \le \frac{\gamma}{(R-r)^2} \int_{B_R} (u-t)_+^2\,, \qquad \forall B_r \subset\subset B_R \subset\subset \Omega\,, \forall t \in \mathbb{R}\,; \tag{3.3}$$

u is in the **lower De Giorgi class** $DG_\gamma^-(\Omega)$ if

$$\int_{\{u<t\}\cap B_r} |\nabla u|^2 \le \frac{\gamma}{(R-r)^2} \int_{B_R} (t-u)_+^2\,, \qquad \forall B_r \subset\subset B_R \subset\subset \Omega\,, \forall t \in \mathbb{R}\,. \tag{3.4}$$

Finally, we set $DG_\gamma(\Omega) = DG_\gamma^+(\Omega) \cap DG_\gamma^-(\Omega)$. We can thus reformulate our final goal as follows:

Theorem 3.1 (De Giorgi's theorem)**.** *If $u \in DG_\gamma(\Omega)$ for some $\gamma > 0$, then $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ for some $\alpha = \alpha(n,\gamma) \in (0,1)$.*

It is a fact that solutions (or functions related to solutions) of many other PDE and/or variational problems belong to a De Giorgi's class, or at least to some closely related notion of regularity class for which one can still prove Theorem 3.1. This versatility has been of course crucial in establishing the relevance of these methods.

Coming back to the goal of this section, we turn to the problem of showing the boundedness of $u \in DG(\Omega)$. We shall actually focus on proving *upper bounds* for $u \in DG_\gamma^+(\Omega)$ (lower bounds for $u \in DG_\gamma^-(\Omega)$ will follow immediately by noticing that $v \in DG_\gamma^+$ iff $-v \in DG_\gamma^-$).

In order to prove upper bounds, we look back to the previous section. There the unilateral, uniform bound $E \subset \{x_1 < t_0 + T_0\}$ was obtained as a consequence of a decay estimate for the decreasing "control function" $V(t) = |E \cap \{x_1 < t\}|$. In the case of u, we shall need to introduce two control functions (this is technical, and somehow not really necessary, see the introduction of $\Phi = V\,I^p$ in the proof below) depending monotonically on two independent variables, the localization radius r and the truncation height t: the functions are

$$V(r,t) = |B_r \cap \{u > t\}|\,, \qquad I(r,t) = \int_{B_r} (u-t)_+^2\,.$$

Notice that they are both increasing in r and decreasing in r. If either V or I vanishes at some (r,t), this means that u is bounded from above by t inside B_r. The fact that two parameters are needed, rather than just one, is related to the fact that if u solves (1.1), then for every $a > 0$ and $b \in \mathbb{R}$, $v(x) = u(a\,x) + b$ still solves an elliptic equation with same ellipticity constants (similarly, if $u \in DG_\gamma^+(\Omega)$, then automatically $v \in DG_\gamma^+(\Omega/a)$).

With this premise in mind, we state the main result of this section.

Theorem 3.2. *If $u \in DG_\gamma^+(\Omega)$, then for every $B_R \subset\subset \Omega$ and every $t \in \mathbb{R}$ we have*

$$\mathcal{L}^n \sup_{B_{R/2}} u \le t + C(n,\gamma)\Big(\frac{I(R,t)}{R^n}\Big)^{1/2}\Big(\frac{V(R,t)}{R^n}\Big)^{\varepsilon/2} \tag{3.5}$$

where

$$\varepsilon = \sqrt{\frac{1}{4}+\frac{2}{n}} - \frac{1}{2}\,. \tag{3.6}$$

Recall that in the previous section we proved that if E is an isoperimetric set, or actually any set satisfying (2.9) for some constant $C(n)$, and if $V(t_0) \le \varepsilon_0 = \varepsilon_0(n,|E|,|E|/P(E))$ as in (2.12), then $E \subset \{x_1 < t_0 + T_0\}$ with $T_0 = 4\,n\,V(t_0)$. The inclusion $E \subset \{x_1 < t_0 + T_0\}$ should be compares to (3.5), which is indeed an inclusion (of the subgraph of u into a horizontal half-space in $\mathbb{R}^{n+1}$).

Proof. Step one: A key idea section 2 was combining our basic minimality inequality (2.9), in its alternative form (2.10), with the (non-sharp) isoperimetric inequality to obtain (2.11). Indeed, this allowed us to obtain a sublinear power of $V(t)$ on the LHS of (2.11), which, ultimately, was responsible for the geometric decay rate of $V(t)$, and thus for the boundedness result. We now mimic that argument, with the (non-sharp) L^2-Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} v^{2^\star}\Big)^{1/2^\star} \le C(n)\Big(\int_{\mathbb{R}^n}|\nabla v|^2\Big)^{1/2}\,, \qquad 2^\star = \frac{2n}{n-2}\,, \tag{3.7}$$

playing the role of the (non-sharp) isoperimetric inequality. As a result, we prove that

$$I(r,t) \le \frac{C(n,\gamma)}{(R-r)^2}\, I(R,t)\, V(r,t)^{2/n}\,, \qquad \forall r < R\,, t \in \mathbb{R}\,, \tag{3.8}$$

where the $2/n$-power descends directly from the Sobolev exponent $2^\star$. To prove (3.8), we pick ζ a cut-off function between B_r and $B_{(R+r)/2}$ and apply first Hölder's inequality, and then the Sobolev inequality on $\mathbb{R}^n$ to $\zeta^2(u-t)_+$, to get

$$\begin{aligned} I(r,t) &= \int_{B_r}(u-t)_+^2 \le \int_{\mathbb{R}^n}(\zeta^2(u-t)_+)^2 \\ &\le \Big(\int_{\mathbb{R}^n}(\zeta^2(u-t)_+)^{2^\star}\Big)^{2/2^\star} |\mathrm{spt}(\zeta^2(u-t)_+)|^{1-(2/2^\star)} \\ &\le C(n)\, V(R,t)^{2/n}\int_{\mathbb{R}^n}|\nabla(\zeta^2(u-t)_+)|^2\,, \end{aligned}$$

where, thanks to $|\nabla\zeta| \le C/(R-r)$,

$$\begin{aligned} \int_{\mathbb{R}^n}|\nabla(\zeta^2(u-t)_+)|^2 &\le C\int_{B_{(r+R)/2}\cap\{u>t\}}|\nabla u|^2 + \frac{C}{(R-r)^2}\int_{B_{(R+r)/2}}(u-t)_+^2 \\ &\le \frac{C(n,\gamma)}{(R-r)^2}\int_{B_R}(u-t)_+^2\,. \end{aligned}$$

This proves (3.8). Since both I and V appear in (3.8), in order to iterate (3.8) it will be necessary to also have a control of V in terms of I. This is easily obtained in the form

$$V(r,t) \le \frac{I(r,s)}{(t-s)^2} \qquad \forall s < t\,, r > 0\,. \tag{3.9}$$

To prove (3.9) we just notice that

$$(t-s)^2\, V(r,t) \le \int_{B_r\cap\{u>t\}}(u-s)_+^2 \le I(r,s)\,.$$

Step two: The idea is now to iterate (3.8) and (3.9) to prove decay to zero of the sup-norm in finite space (radius of the ball) and cutting height. (In analogy with section 2, iteration is taking here the role of integration of the ODE obtained there!) In other words, starting from (3.8) and (3.9), we want to show that either $V(R/2, t+h)$ or $I(R/2, t+h)$ is zero for a suitable choice of h depending on $I(R,t)$ and $V(R,t)$ (cf. with (3.5)). From a technical viewpoint, the iteration scheme is much more transparent if rather than looking at V and I through separate chains of inequalities, we rather mix them into a single function of the form

$$\Phi(r,t) = V(r,t)\, I(r,t)^p \qquad \text{for some } p > 0\,.$$

Indeed, $\Phi(R/2, t+h) = 0$ still implies that u is bounded from above by $t+h$ in $B_{R/2}$. The choice of p has to be made to get a neat decay inequality in place of the system of inequalities (3.8)–(3.9). To this end given $r < R$ and $t > s$ we exploit (3.8), (3.9) and the monotonicities of I and V in their variables to get

$$\begin{aligned}\Phi(r,t) = V(r,t)\, I(r,t)^p \ &\leq\ \frac{C(n,\gamma)}{(R-r)^{2p}(t-s)^2}\, V(R,s)^{2p/n}\, I(R,s)^{1+p} \\ &=\ \frac{C(n,\gamma)}{(R-r)^{2p}(t-s)^2}\, \Phi(R,s)^q\end{aligned}$$

for some $q > 0$ provided n and p are related by

$$1 + p = qp\,, \qquad q = \frac{2p}{n}\,.$$

We thus obtain

$$\frac{2}{n}p^2 - p - 1 = 0 \qquad \text{iff} \qquad p = \frac{n}{4} + \frac{n}{4}\sqrt{1 + \frac{8}{n}}\,,$$

which gives

$$q = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{n}} = 1 + \varepsilon \qquad \varepsilon = \sqrt{\frac{1}{4} + \frac{2}{n}} - 1 > 0\,.$$

We thus have

$$\Phi(r,t) \leq \frac{C(n,\gamma)}{(R-r)^{2p}(t-s)^2}\, \Phi(R,s)^{1+\varepsilon}\,, \qquad \forall r < R\,, t > s\,. \tag{3.10}$$

We now fix $R_0 = R$, $t_0 = t$ and set

$$R_k = \frac{R}{2} + \frac{R}{2^k}\,, \qquad t_k = t + h - \frac{h}{2^k}\,,$$

so that $R_0 = R$, $R_\infty = R/2$, $t_0 = t$, and $t_\infty = t+h$. By iterating (3.10) we can thus bound $\Phi(R/2, t+h)$ from above in terms of $\Phi(R, t+d)$. To see how this work, we just notice that by (3.10) we have

$$\Phi(R_{k+1}, t_{k+1}) \leq \frac{C(n,\gamma)}{R^{2p}\, h^2}\, 2^{(k+1)2p}\, 2^{(k+1)2}\, \Phi(R_k, t_k)^{1+\varepsilon} = \frac{C(n,\gamma)}{R^{2p}\, h^2}\, 2^{k(2p+2)}\, \Phi(R_k, t_k)^{1+\varepsilon} \tag{3.11}$$

The idea now is that the Φ^ε term on the RHS is enough to dampen out the exponential factor $2^{k(2p+2)}$. This is more clearly checked by rewriting (3.11) in terms of $\Psi_k = \Phi(R_k, t_k)\, 2^{k\,\mu}$ for $\mu > 0$ to be chosen,

$$\begin{aligned}\psi_{k+1} \ &\leq\ \frac{C(n,\gamma)}{R^{2p}\, h^2}\, 2^{k(2p+2)}\, 2^{(k+1)\,\mu - (1+\varepsilon)\,k\,\mu}\, \psi_k^{1+\varepsilon} \\ &=\ \frac{C(n,\gamma)\, 2^\mu}{R^{2p}\, h^2}\, 2^{k(2p+2-\varepsilon\,\mu)}\, \psi_k^{1+\varepsilon}\,.\end{aligned} \tag{3.12}$$

Hence the choice of μ is dictated by (3.12), to be

$$\mu = \frac{2p+2}{\varepsilon}\,.$$

In this way (3.12) becomes

$$\psi_{k+1} \le \frac{C(n,\gamma)\,2^\mu}{R^{2p}\,h^2}\,\psi_k^{1+\varepsilon}\,, \qquad \forall k\,, \tag{3.13}$$

which is what is needed to show that if ψ_0 is small enough, then $\psi_k \le \psi_0$ for every k. Indeed, by (3.13), in order to have $\psi_1 \le \psi_0$ we just need to choose h so that

$$\frac{C(n,\gamma)\,2^\mu}{R^{2p}\,h^2}\,\psi_0^{\varepsilon} = 1\,. \tag{3.14}$$

Arguing inductively, if we have already proved that $\psi_k \le \psi_0$, we deduce from (3.14) that

$$\frac{C(n,\gamma)\,2^\mu}{R^{2p}\,h^2}\,\psi_k^{\varepsilon} \le 1\,,$$

and thus from (3.13) that $\psi_{k+1} \le \psi_0$. Since $\psi_k \le \psi_0$ for every k implies $\Phi(R_k,t_k) \le 2^{-\mu\,k}\,\Phi(R_0,t_0)$ for every k we conclude that $0 = \Phi(R_\infty,t_\infty) = \Phi(R/2,t+h)$, and thus

$$\mathcal{L}^n \sup_{B_{R/2}} u \le t+h\,, \tag{3.15}$$

with h as in (3.14). Using $1+p=(1+\varepsilon)p$ and $\frac{2}{n}\,p^2-p-1=0$ we see that (3.15) is equivalent to (3.5). □

4. A criterion for Hölder continuity and decay of level sets

Lebesgue points theorem implies that if $u \in L^p_{\rm loc}(\Omega)$, $1 \le p < \infty$, then for a.e. $x \in \Omega$

$$\lim_{r\to 0^+}\frac{1}{r^n}\int_{B_r(x)}|u-(u)_{x,r}|^p = 0\,, \qquad \text{where} \quad (u)_{x,r} = \frac{\int_{B_r(x)} u}{|B_r(x)|}\,.$$

If the rate of this decay to zero can be quantified uniformly on $x \in \Omega$ by a power r^α for some $\alpha \in (0,1]$, then u is equivalent to a $C^{0,\alpha}_{\rm loc}(\Omega)$ function. More formally, Campanato's criterion states that the existence of positive constants K and r_0 such that if

$$\int_{B_r(x)}|u-(u)_{x,r}|^p \le K\,r^{n+\alpha\,p}$$

for every $x \in \Omega$ and every $r < \min\{{\rm dist}(x,\partial\Omega),r_0\}$, then $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ (modulo an a.e. modification). When $u \in L^\infty_{\rm loc}(\Omega)$, we can define the oscillation of u in $B_r(x)$ as

$$\omega_x(r) = \mathcal{L}^n \sup_{B_r(x)} u - \mathcal{L}^n \inf_{B_r(x)} = M_x(r) - m_x(r)\,,$$

and easily deduce from Campanato's criterion that, if

$$\omega_x(r) \le K\,r^\alpha\,, \qquad \text{for every } x \in \Omega \text{ and } r < \min\{{\rm dist}(x,\partial\Omega),r_0\}\,, \tag{4.1}$$

then u is equivalent to a $C^{0,\alpha}_{\rm loc}(\Omega)$ function. In turn, a practical way to check the validity of (4.1) is given in the following lemma (notice that $\omega_x(r)$ is defined for $r < {\rm dist}(x,\partial\Omega)$ and is increasing in r).

Lemma 4.1. *If $\omega:(0,r_0)\to[0,\infty)$ is an increasing function such that there exist $t,\eta \in (0,1)$ with*

$$\omega(t\,r) \le \eta\,\omega(r)\,, \qquad \forall r \in (0,r_0)\,,$$

then $\omega(r) \le C\,r^\alpha$ for every $r \in (0,r_0)$ where

$$\alpha = \frac{\log\eta}{\log t}\,, \qquad C = \frac{\omega(r_0)}{\eta\,r_0^\alpha}\,.$$

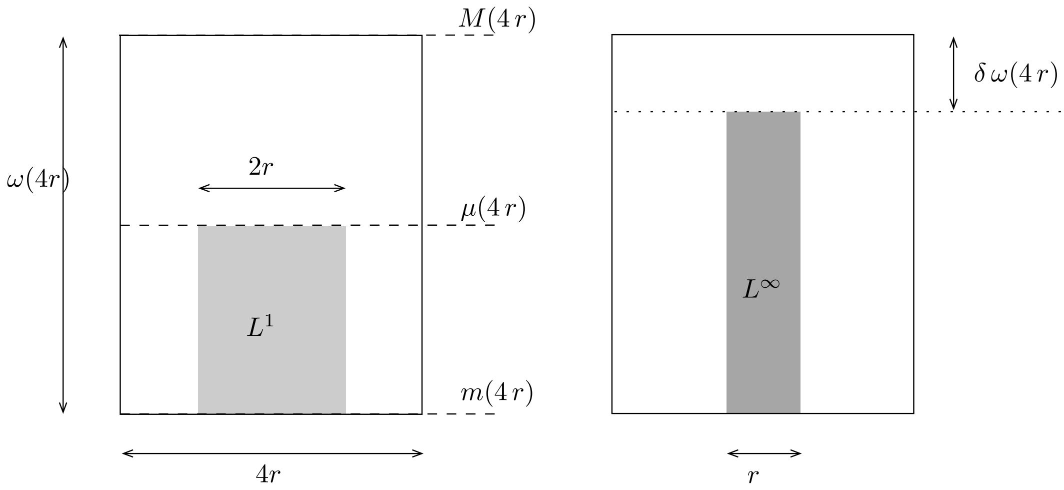

FIGURE 4.1. A schematic representation of the assumption (on the left) and of the conclusion (on the right) of Theorem 4.2, where a (sort of) L^1-bound is shown to imply a one-sided L^∞-bound. The diagram on the left is intended to suggest that u "takes most of its values on B_{2r}" below its average $\mu(4r)$ on B_{4r}; the precise form of this concept is (4.2). Theorem 4.2 states that if (4.2) holds, then we have the diagram on the right, i.e., u takes *all* of its values below $M(4r) - \delta\,\omega(4r)$.

Proof. Let $r \in (0, r_0)$ and let $k \in \mathbb{N}$ be such that $t^{k+1}\, r_0 \le r < t^k\, r_0$, then

$$\begin{aligned} \omega(r) \;&\le\; \omega(t^k\, r_0) \le \eta^k\, \omega(r_0) = \frac{\omega(r_0)}{\eta}\, \eta^{k+1} \\ &= \frac{\omega(r_0)}{\eta}\, t^{(k+1)\alpha} = \frac{\omega(r_0)}{\eta\, r_0^\alpha}\, (t^{k+1}\, r_0)^\alpha \le \frac{\omega(r_0)}{\eta\, r_0^\alpha}\, r^\alpha\,. \end{aligned}$$

□

Thus the goal is showing the existence of $t, \eta \in (0,1)$ and $r_0 > 0$ such that

$$\omega_x(\eta\, r) \le t\, \omega_x(r)$$

whenever $x \in \Omega$ and $r < \min\{\mathrm{dist}(x, \partial\Omega), r_0\}$. We drop the dependency on x and set

$$B_r = B_r(x)\,, \qquad m(r) = \mathcal{L}^n \inf_{B_r} u\,, \qquad M(r) = \mathcal{L}^n \sup_{B_r} u\,,$$

$$\omega(r) = M(r) - m(r)\,, \qquad \mu(r) = \frac{M(r) + m(r)}{2}\,.$$

We shall now prove the existence of $\delta \in (0,1)$ such that if $B_{4r} \subset\subset \Omega$, then

$$\omega(r) \le (1-\delta)\, \omega(4r)\,.$$

(The change of variables $\eta = 1 - \delta$ is adopted because it leads to nicer formulas.) Notice that $\omega(r)$ may be smaller than $\omega(4r)$ either because $M(r)$ has decreased by $-\delta\,\omega(4r)$ with respect to $M(4r)$, or because $m(r)$ has increased by $\delta\,\omega(4r)$ with respect to $m(4r)$, in other words, it is enough to show that if $B_{4r} \subset\subset \Omega$, then either

$$M(r) \le M(4r) - \delta\, \omega(4r)$$

or

$$m(r) \ge m(4r) + \delta\, \omega(4r)\,.$$

An important insight is that one can decide which of the two alternatives is going to happen by looking at the relative size of $V(2\,r, \mu(4r)) = |B_{2r} \cap \{u > \mu(4r)\}|$ with respect to $|B_{2r}|$. In other words, see Figure 4.1, by looking to the volume fraction of B_{2r} where u

is above its B_{4r}-average, we can tell if it is its B_r-maximum that has to decrease, or if it is its B_r-minimum that has to increase.

Theorem 4.2 (De Giorgi's decay theorem)**.** *Given $n \ge 3$ and $\gamma > 0$ there exists $\delta \in (0,1)$ with the following property. If $u \in DG^+_\gamma(\Omega)$, $B_{4r} \subset\subset \Omega$ and*

$$\frac{V(2\,r,\mu(4\,r))}{|B_{2r}|} \le \frac{1}{2}\,, \tag{4.2}$$

then $M(r) \le M(4r) - \delta\,\omega(4r)$.

Proof. Step one: We set

$$t_0 = \mu(4r)\,, \qquad t_k = M(4r) - \frac{\omega(4r)}{2^{k+1}}\,, \qquad t_\infty = M(4r)\,,$$

and show that

$$\frac{V(2\,r,t_k)}{|B_{2r}|} \le \frac{C(n,\gamma)}{k^{n/2(n-1)}}\,, \qquad \forall k \ge 1\,. \tag{4.3}$$

This estimate may look strange, since $V(2r,t) = 0$ for every $t \in (M(2r), M(4r))$ and $t_k \in (M(2r), M(4r))$ every k large enough. In other words, (4.3) is trivial for infinitely many values of k! The real interest of this estimate is that it is quantitative in k, so that it allows to conclude that $V(2\,r,t_k)$ is below any prescribed fraction λ of $|B_{2\,r}|$ provided k is chosen large enough in terms of λ (and n and γ).

This decay estimate for V is proved by applying the Sobolev inequality jointly with $u \in DG^+_\gamma(\Omega)$ to a double truncation of u. The Sobolev inequality we shall need to use is the following: if $v \in W^{1,1}(B_{2r})$, $v \ge 0$, and $|\{v = 0\} \cap B_{2r}| \ge |B_{2r}|/2$, then

$$\Big(\int_{B_{2r}} v^{n/(n-1)}\Big)^{(n-1)/n} \le C(n) \int_{B_{2r}} |\nabla v|\,. \tag{4.4}$$

Given $s < t$, we apply this inequality to

$$v = \min\{t,u\} - \min\{s,u\} = \begin{cases} t - s\,, & \text{on } \{u > t\}\,, \\ u - s\,, & \text{on } \{t \ge u > s\}\,, \\ 0\,, & \text{on } \{u \le s\}\,, \end{cases}$$

provided

$$\frac{|\{u < s\} \cap B_{2r}|}{|B_{2r}|} \ge \frac{1}{2}\,.$$

Of course this is our case, by (4.2), if we take $s \ge \mu(4r)$. We thus find

$$\begin{aligned}
(t-s)\,V(2\,r,t)^{(n-1)/n} &\le \Big(\int_{B_{2r}} v^{n/(n-1)}\Big)^{(n-1)/n} \le C(n) \int_{B_{2r}} |\nabla v| \\
&= C(n) \int_{\{s<u\le t\}\cap B_{2r}} |\nabla u| \\
&\le C(n) \Big(\int_{\{s<u\le t\}\cap B_{2r}} |\nabla u|^2\Big)^{1/2} \Big(V(2\,r,s) - V(2\,r,t)\Big)^{1/2} \\
&\le \frac{C(n,\gamma)}{r} \Big(\int_{B_{4r}} (u-s)_+^2\Big)^{1/2} \Big(V(2\,r,s) - V(2\,r,t)\Big)^{1/2} \\
&\le C(n,\gamma)\,(M(4\,r) - s)_+\, r^{(n/2)-1} \Big(V(2\,r,s) - V(2\,r,t)\Big)^{1/2}\,,
\end{aligned}$$

that is

$$V(2\,r,t)^{2(n-1)/n} \le C(n,\gamma)\,\frac{(M(4\,r)-s)_+^2}{(t-s)^2} \Big(V(2\,r,s) - V(2\,r,t)\Big)\, r^{n-2}\,.$$

We plug in

$$t = t_{k+1} = M(4\,r) - \frac{\omega(4\,r)}{2^{k+2}}\,, \qquad s = t_k = M(4\,r) - \frac{\omega(4\,r)}{2^{k+1}} \geq t_0 = \mu(4\,r)\,,$$

and get

$$V(2\,r, t_{k+1})^{2(n-1)/n} \leq C(n,\gamma)\,\frac{2^{2(k+2)}}{2^{2(k+1)}}\Big(V(2\,r,t_k) - V(2\,r,t_{k+1})\Big)\,r^{n-2}\,,$$

so that

$$\begin{aligned}(N+1)\,V(2\,r,t_{N+1})^{2(n-1)/n} &\leq \sum_{k=0}^{N} V(2\,r,t_{k+1})^{2(n-1)/n}\\ &\leq C(n,\gamma)\,r^{n-2}\sum_{k=0}^{N}\Big(V(2\,r,t_k) - V(2\,r,t_{k+1})\Big)\\ &\leq C(n,\gamma)\,r^{n-2}\,V(2\,r,t_0) \leq C(n,\gamma)\,r^{2n-2}\,,\end{aligned}$$

that is

$$\Big(\frac{V(2\,r,t_{N+1})}{r^n}\Big)^{2(n-1)/n} \leq \frac{C(n,\gamma)}{N}\,,$$

as desired.

Step two: We now complete the proof of the theorem. By Theorem 3.2, for every $t \in R$ we have

$$M(r) \leq t + C(n,\gamma)\,\Big(\frac{I(2\,r,t)}{r^n}\Big)^{1/2}\,\Big(\frac{V(2\,r,t)}{r^n}\Big)^{\varepsilon/2} \tag{4.5}$$

where $\varepsilon = \varepsilon(n) > 0$ is as in (3.6). The goal is showing $M(r) \leq M(4\,r) - \delta\,\omega(4\,r)$, so we set as in step one

$$t_0 = \mu(4r)\,, \qquad t_k = M(4r) - \frac{\omega(4r)}{2^{k+1}}\,, \qquad t_\infty = M(4r)\,,$$

and apply (4.6) with t_k for some k to be chosen: we thus find

$$M(r) \leq M(4\,r) - \frac{\omega(4r)}{2^{k+1}} + C(n,\gamma)\,\Big(\frac{I(2\,r,t_k)}{r^n}\Big)^{1/2}\,\Big(\frac{V(2\,r,t_k)}{r^n}\Big)^{\varepsilon/2}\,. \tag{4.6}$$

Clearly

$$\frac{I(2\,r,t_k)}{r^n} = \frac{1}{r^n}\int_{B_{2r}}(u-t_k)_+^2 \leq C(n)\,(M(4\,r)-t_k)^2 \leq C(n)\,\Big(\frac{\omega(4\,r)}{2^{k+1}}\Big)^2\,.$$

while the uniform decay estimate (4.3) gives

$$\frac{V(2\,r,t_k)}{r^n} \leq \frac{C(n,\gamma)}{k^{n/2(n-1)}}$$

so that (4.6) implies

$$M(r) \leq M(4\,r) - \frac{\omega(4r)}{2^{k+1}} + \frac{C_*(n,\gamma)}{k^{\varepsilon\,n/4(n-1)}}\,\frac{\omega(4\,r)}{2^{k+1}}\,. \tag{4.7}$$

Let k_0 be such that

$$\frac{C_*(n,\gamma)}{k_0^{\varepsilon\,n/4(n-1)}} \leq \frac{1}{2}\,.$$

Then (4.7) gives

$$M(r) \leq M(4\,r) - \frac{\omega(4r)}{2^{k_0+1}}\,,$$

and the theorem is proved with $\delta = 2^{-(k_0+1)}$. □

Proof of De Giorgi's theorem, Theorem 3.1. If $u \in DG_\gamma(\Omega)$, then Theorem 3.2 can be applied to both u and $-u$ to prove that $u \in L^\infty_{\rm loc}(\Omega)$. In particular, we can define $M_x(r)$, $m_x(r)$, $\omega_x(r)$ and $\mu_x(r)$ for every $B_r(x) \subset\subset \Omega$. Fix $B_r = B_r(x)$ with $B_{4r} \subset\subset \Omega$, and set $M = M_x$, $m = m_x$, etc. Then, either

$$\frac{V(2\,r,\mu(4\,r))}{|B_{2\,r}|} \le \frac{1}{2}\,,$$

or

$$\frac{V(2\,r,\mu(4\,r))}{|B_{2\,r}|} > \frac{1}{2}\,.$$

In the first case, $M(r) \le M(4\,r) - \delta\,\omega(4\,r)$ with $\delta = \delta(n,\gamma)$ as in De Giorgi's decay theorem; in the second case, we can apply De Giorgi's decay theorem to $-u$ and find $m(r) \ge m(4\,r) + \delta\,\omega(4\,r)$. In both cases we have proved that $\omega(r) \le (1-\delta)\,\omega(4\,r)$ whenever $B_{4\,r} \subset\subset \Omega$. This implies that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ with $\alpha = \log(1-\delta)/\log(1/4) = \log_4(1/(1-\delta))$. □

References

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Department of Mathematics, The University of Texas at Austin, 2515 Speedway, Stop C1200, Austin TX 78712-1202, USA

E-mail address: maggi@math.utexas.edu